Large-Scale Networks

4 - Structures
Introduction

During the previous lecture, we looked at network evolution through:

- Selection
- Social influence

This week, we will talk about relationships that affect the structures:

- Positive relationships
- Negative relationships
Roadmap

– Structural Balance

– Applications

– Weak structural balance

– Generalization
Structural Balance
Structural balance

- The theory behind structural balance comes from social psychology [Hei40]

- Take two connected persons in isolation
  - Label the edge + if they are friends
  - Label the edge – if they are enemies
  \[\Rightarrow\] A *signed graph* is a graph in which each edge has a positive or a negative sign

- Take three connected persons, certain configurations are more plausible than others
Structural balance

- Take three persons, A, B and C connected to each other
- What kind of configurations can we have?

1. 3 pluses:
   - This is a natural situation
   - It corresponds to three people who are mutually friends

A, B and C are mutual friends
Structural balance

- Take three persons, A, B and C connected to each other
- What kind of configurations can we have?

2. 1 plus and 2 minuses:
   - This is a natural situation
   - Two of the three are friends and they have a mutual enemy

A and B are friends and have C as a mutual enemy: balanced
Structural balance

- Take three persons, A, B and C connected to each other
- What kind of configurations can we have?

3. 2 pluses and 1 minus:
   - Creates some instability
   - A is friends with B and C who do not get along with each other

![Diagram showing A is friends with B and C but they are not friends: not balanced]
Structural balance

- Take three persons, A, B and C connected to each other
- What kind of configurations can we have?

4. 3 minuses:
   - Creates some instability
   - There are forces motivating two persons to team up against the third

A, B and C are mutual enemies: not balanced
Structural balance

- Conclusions
  - We refer to triangles with one or three ‘+’ as \textit{balanced} since they are \textit{free} from instability
  - We refer to triangles with zero or two ‘+’ as \textit{unbalanced} since they are unstable

- \textbf{Unbalanced} triangles are sources of \textit{stress} so that people strive to minimize them in their personal relationships
  \[\Rightarrow\] \textbf{Unbalanced} triangles will thus be \textit{less abundant} in real social settings than \textit{balanced} triangles
Structural balance

- How to generalize structural balance to any complete graph?

- A labeled complete graph is balanced if every one of its triangles is balanced

**Structural balance property (SBP):** For every set of three nodes, if we consider the three edges connecting them, either all three of these edges are labeled $+$ or exactly one of them is labeled $+$. 
Structural balance

- Examples:
  - The labeled four-node complete graph on the left is balanced because each set of 3 nodes satisfies the structural balance property.
  - The one on the right is unbalanced because triangles A, B, C and B, C, D violate the structural balance property.
Structural balance

- At a higher level, how does a balance network look like?

- One way to be balanced, is if everyone likes each other
  - All triangles have thus three ‘+’ labels

- A slightly more complicated representation would be:
  - Consider two groups X and Y
  - Everyone in X likes each other
  - Everyone in Y likes each other
  - And everyone in X is the enemy of everyone in Y
Structural balance

- If a complete graph can be divided into two sets of mutual friends, with complete mutual antagonism between the two sets, then it is balanced. This is actually the only way for a complete graph to be balanced.
Structural balance

- This leads to two basic ways of achieving structural balance:
  - Everyone likes each other
  - The world consists of two groups of mutual friends with complete antagonism between the groups

The Balance Theorem [Hav53]: If a labeled complete graph is balanced, then either all pairs of nodes are friends, or else the nodes can be divided into two groups, X and Y, such that each pair of people in X likes each other, each pair of people in Y likes each other and everyone in X is the enemy of everyone in Y

- The balance theorem takes a local property (structural balance property) and implies a global property:
  - either everyone gets along
  - or the world is divided into two enemy groups
Structural balance

Proof
- Suppose we have a balanced arbitrary labeled complete graph
- If the graph has only + labels, then we are done. Assume this isn’t the case

  - Let A be a node of a group X and let Y be another group
    - Every other node is either a friend of A or an enemy of A (due to completeness)
      - Let X be A and all its friends and Y be the rest
  - We need to show 3 properties:
    1. Every two nodes in X are friends
    2. Every two nodes in Y are friends
    3. Every node in X is an enemy of every node in Y
- We now show that our definition of X and Y satisfies these 3 properties
- **Proof (con’t):** A schematic illustration of our analysis of balanced networks. (There may be other nodes not illustrated here.)
Structural balance

- **Proof (con’t)**

- We now show that our definition of $X$ and $Y$ satisfies these 3 properties

1. $A$ is friends with every other node in $X$
   \[ \Rightarrow B \text{ and } C \text{ in } X \text{ are friends as well, otherwise triangle } A, B, C \text{ would violate SBP} \]

2. $A$ is enemy with every node in $Y$
   \[ \Rightarrow D \text{ and } E \text{ in } Y \text{ are friends, otherwise triangle } A, D, E \text{ would violate SBP} \]

3. $A$ is friend with any $B$ in $X$ and enemy with any $D$ in $Y$
   \[ \Rightarrow B \text{ and } D \text{ are enemies, otherwise triangle } A, B, D \text{ would violate SBP} \]

- This concludes the proof
Applications
Applications

Let’s consider two type of applications of structural balance

1. **International relations** can be represented as a network of countries whose relations are a combinations of alliances and animosities

2. **Online rating Web sites** offer individuals the possibility to express positive or negative opinions about each other
International relations

- International relations is a setting in which it is natural to assume that a collection of nodes all have opinions (positive or negative) about one another
  - Nodes are nations
  - Edge labeled + indicate alliance
  - Edge labeled – indicate animosity

- Structural balance sometimes explains behaviors of nations during crises
  - Conflict between Bangladesh’s separation and Pakistan in 1972
  - US support to Pakistan is not surprising considering that:
    • USSR (R) was China’s enemy
    • China was India’s enemy
    • India had bad relations with Pakistan [Mor78]
International relations

- The shifting alliances preceding World War I was used as another example of structural balance in international relations [AKR06]

(a) Three Emperors' league, 1872-1881

(b) Triple alliance, 1882

(c) German-Russian Lapse, 1890

- The evolution of alliances in Europe, 1872-1907 (the nations GB, Fr, Ru, It, Ge, and AH are Great Britain, France, Russia, Italy, Germany, and Austria-Hungary respectively). Solid dark edges indicate friendship while dotted red edges indicate enmity.
International relations

- Note how the network slides into a balanced labeling — and into World War I [AKR06]

(d) French Russian Alliance, 1892-1917

(e) Entente Cordiale 1904

(f) British Russian alliance, 1907
Online ratings

- Slashdot is a news website on science and technology
  - users can designate each other as a “friend” or a “foe”

- Epinions is an online product rating site
  - Users evaluate products
  - Users express trust or distrust of other users
Online ratings

- Epinion analysis revealed differences between online ratings and friend-enemy dichotomy of structural balance theory [GKR+04]
  - Users of Epinion form a directed graph
  - If A trusts B and B trusts C then A should trust C
  - If A distrusts B and B distrusts C then should A distrust C?
    - If distrust was like enmity then yes
    - If someone distrust someone else because she is more knowledgeable then we should expect the opposite
Online ratings

- Conclusions
  - Understanding the implications of positive and negative relationships is not easy
  - However, this is necessary to measure the role they play on social Web sites where users register subjective evaluations of each other
  - How to apply the theory of structural balance in these large-scale networks is a recent research question

- Let’s try now to refine our theory of structural balance to make it more realistic by weakening our notion of structural balance
Weak Structural Balance
Weak structural balance

- There are two configurations in which triangles were considered unbalanced
  - 2 pluses (someone trying to reconcile his friends)
  - 0 plus (two nodes will ally themselves against the other)

- Some studies report that the former case is stronger than the later [Dav67]

- It becomes natural to ask what structural properties arise when we rule out only triangles with exactly two positive edges
Weak structural balance

Weak structural balance property: There is no set of three nodes such that the edges among them consist of exactly two positive edges and one negative edge.

- There should be more weakly balanced networks than balanced networks under the former definition

- Actually, there are new kinds of structures that can arise in weakly balanced networks
Weak structural balance

- A complete graph is **weakly balanced** precisely when it can be divided into multiple sets of mutual friends, with complete mutual antagonism occurring between each pair of sets.
Weak structural balance

- We can easily check the weak structural balance property of a network
  - Suppose that nodes can be divided into groups:
    - Two nodes are friends if they belong to the same group
    - Two nodes are enemies when they belong to different groups
  - In any triangle that contains at least two “+”, all three nodes must belong to the same group
    \[ \Rightarrow \text{the network contains no triangles with exactly two “+” edges} \]

Characterization of Weakly Balanced Networks: If a labeled complete graph is weakly balanced, then its nodes can be divided into groups in such a way that every two nodes belonging to the same group are friends, and every two nodes belonging to different groups are enemies.

- Weakly balanced networks can contain any number of opposed groups of mutual friends
Generalization
Generalization

- So far our definitions are restrictive:
  - They only apply to complete graph
    - However, persons may not have an opinion on others

- Can we relax the balance theorem so that if most triangles are balanced we can still approximately divide the world into two factions?
Generalization

Let’s consider a social network that is not necessarily complete

- Two nodes may be linked by a positive edge
- Two nodes may be linked by a negative edge
- Two node may not be linked to each other

We can relax the former definition in two ways:

1. We consider that the given graph misses some edges:

   The network is balanced if we can complete it with some edges that lead to a complete graph that is balanced under the former definition

2. We consider that the given graph should be divisible into two sets:

   The network is balanced if it is possible to divide the nodes into two sets, so that any edge within one set is positive, any edge across sets is negative
Generalization

Examples

1. A graph can be completed into a complete graph that satisfies the former property.

2. A graph can be divided into two sets with positive intra-set and negative inter-set edges.
Generalization

These two definitions (by completing edges or dividing nodes) are equivalent

- Definition (1) implies definition (2)
  - If a signed graph is balanced under the definition (1) then after filling in all the missing edges appropriately we obtain a signed complete graph where we can apply the Balance Theorem
  - This approach divides the network into two sets, X and Y, that satisfy the properties of the definition (2)

- Definition (2) implies definition (1)
  - If a signed graph is balanced under definition (2) then after finding a division of the nodes into X and Y, we can fill positive edges inside X and inside Y and fill in negative edges between X and Y and check that all triangles will be balanced, satisfying definition (1)
Generalization

- Is this graph balanced?
Generalization

- No it is not balanced
  - Try going through each edge clock-wise
  - Place endpoints in the same set if you cross a + edge
  - Place them in different sets if you cross a – edge
  - You cannot do that for all edges without changing your initial decision

- Getting back to node 1 induced crossing an odd number of negative edges
Claim: A signed graph is balanced if and only if it contains no cycle with an odd number of negative edges.

- The proof proceeds by
  - either finding a balanced division in sets X and Y in which all edges are positive and across which all edges are negative
  - or finding a cycle with an odd number of negative edges

- Find the supernodes representing blobs of positively connected nodes so that supernodes are connected through negative edges
To determine whether a signed graph is balanced, the first step is to consider only positive edges, to find the supernodes.
Generalization

- If any supernode contains a negative edge between some pair of nodes A and B then the graph contains a cycle with an odd number of negative edges:
  - take a path of positive edges from B to A and
  - take the r
Generalization

- We can thus consider a simpler graph whose nodes are the supernodes of the previous graph and there is an edge between two supernodes if each of them contained a node connected with
Generalization

- A more standard drawing of the previous graph where we visualize the negative edges between supernodes

- From now on, there are two options:
  - Either we label each node in this reduced graph as X or Y so that each edge connects X to Y
  - Or we find a cycle in the reduced graph with an odd number of edges
Generalization

- Once we have found a cycle of an odd number of negative edges in the reduced graph, we can determine a cycle of an odd number of negative edges in the original graph by listing the nodes connected within a supernode with positive edges and that connect

![Diagram of a graph showing nodes and edges with positive and negative signs.](image-url)
Generalization

- This version of finding an “odd” cycle where the underlying graph has only negative edges is known as the problem of determining whether a graph is bipartite
  - Whether its nodes can be divided into two groups (e.g., X and Y) so that each edge goes from one group to the other

- We defined bipartite graphs in the previous lecture while presenting affiliation networks

- If we can find whether the graph is bipartite, then we know whether there are no odd cycles…

- How to determine whether a graph is bipartite using breadth-first search (BFS)?
Generalization

- We start a BFS from any node in the graph (e.g. $G$), producing layers
- Because edges cannot jump over a layer of the breadth-first search, then
  1. Edges connect nodes in adjacent layers or
  2. nodes in the same layer
Generalization

- We start a BFS from any node in the graph (e.g. G), producing layers
- Because edges cannot jump over a layer of the breadth-first search, then
  1. Edges connect nodes in adjacent layers or
  2. nodes in the same layer
- **Case 1**: balanced division
  - even-numbered layers as part of X
  - odd-numbered layers as part of Y

An odd cycle is formed from two equal-length paths leading to an edge inside a single layer.
Generalization

- We start a BFS from any node in the graph (e.g. G), producing layers
- Because edges cannot jump over a layer of the breadth-first search, then
  1. Edges connect nodes in adjacent layers or
  2. nodes in the same layer
- **Case 1: balanced division**
  - even-numbered layers as part of X
  - odd-numbered layers as part of Y
- **Case 2: cycle**
  - two connected nodes (A and B) in the same layer have an immediate common ancestor (D)
  - the length of paths from D to A and from D to B are of same size k
  \[ \Rightarrow \text{This creates a cycle of size } 2k+1: \text{ an odd number} \]
Conclusion
Conclusion

- A signed graph represents the positive and negative relations in a network

- The organization of relations in the network may produce some stress characterized by a violation of the **Structural Balance Property**

- The **Balance Theorem** illustrates how local relations impact globally the network

- A **weaker** form of balance property is needed to consider more realistic environments, where nodes have a partial knowledge of others

- Determining whether a network is balanced can be achieved through a BFS on the supernodes of the network
References


